PERMEABILITY OF POROUS SOLIDS

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An expression has been derived to describe both saturated and unsaturated permeability of porous media in terms of the pore size distribution as obtained from mercury-injection data or water-desorption isotherms. An interaction model has been adopted wherein both pore radius and effective area available for flow have been considered. The permeability values obtained using this expression have been compared with water and gas permeabilities of a variety of porous media. Satisfactory agreement is found between experimental and calculated values over a wide range of permeability.

The flow of fluids through porous materials is of great significance in the fields of industrial chemistry, oil technology and agriculture. In general, it may be stated that the principal interest is in the transport through reactive materials. However, interpretation of transport data is complicated by the fact that at the present time no entirely adequate description of flow through inert granular materials exists. Two main types of approach are at present being used in this problem; the first results in the Kozeny-Carman equation [1] which is derived using the hydraulic radius and tortuosity concepts. It has been assumed that tortuosity may be obtained from electrical resistivity measurements. This approach has been discussed by Wyllie and Spangler [2] and Faris *et al.*[3] The other treatment of this problem is a statistical one developed by Childs and Collis-George [4] and is based on the probability of the continuity of pores in adjacent places within the porous medium. Both these approaches to the problem depend on the pore size distribution of the medium but that of Childs and Collis-George is related only to the pore size distribution, and requires no additional data such as the electrical resistivity of the porous medium filled with a conducting fluid. More recently, Marshall [5] has developed an equation which is essentially similar to that of Childs and Collis-George.

Recently, Fatt [6] has put forward flow concepts wherein the porous medium is likened to a network of capillaries. In this treatment, and the electrical analogue of Probine,[7] the assumed nature of the porous solid is somewhat similar to that used here.

The theory presented here is a further development of the pore interaction model proposed by Childs and Collis-George.[4] The equations presented closely describe saturated and unsaturated permeability as a function of bed porosity, fluid content and pore size distribution.

THEORY

In a porous solid there is point-to-point variation in the volume, area and linear proportions of solid to non-solid. A porous material is envisaged as consisting of solid spheres which interpenetrate each other, separated by spherical pores which also interpenetrate; the solid and pore systems are therefore symmetrical. By using this model, it is possible to arrive at a generalized relationship between the porosity and the cross-sectional area controlling flow in a porous material. When the porosity of an isotropic porous medium is given by ε ml per ml of bed, then the area of pore space exposed in a cut surface of appreciable extent will be ε cm^2 per cm^2. If an interaction model is adopted to include the probability of continuity of pore space within the medium, the pore area resulting from interaction will be between ε and ε^2 cm² per cm² since part of ε itself is the resultant of pore interaction.

If the area resulting from interaction is ε^{2x}, that is, the resultant of ε^x interacting with ε^x to give an *effective area for flow*, then

$$\varepsilon^x > \varepsilon > \varepsilon^{2x} > \varepsilon^2,$$

and since

$$\varepsilon < 1, \quad 0{\cdot}5 < x < 1.$$

Further, ε^x may be regarded as a maximum area and ε^{2x} a minimum pore area. *If* ε^{2x} cm² per cm² were contained in a single plane it would be associated with a maximum area which could be occupied by solid, which by a symmetrical application of the above procedure, would be given by $(1-\varepsilon)^x$ cm² per cm². Hence, the minimum pore area in the absence of interaction is given by $1-(1-\varepsilon)^x$ cm². Both the minimum pore area obtained in this way and the minimum area obtained after interaction should be identical and both limit flow so that

$$\varepsilon^{2x} + (1-\varepsilon)^x - 1 = 0.$$

For values of ε between 0·1 and 0·6, x lies between 0·6 and 0·7 and for simplicity x may be taken as $\frac{2}{3}$.

When a hydrostatic suction or tension of sufficient magnitude is applied to the water contained in a porous medium, water will be withdrawn from the pore space. For a given material there will be a characteristic relationship between water content and applied suction (moisture characteristic[4]). The moisture characteristic may be obtained by placing the saturated porous medium on a suitable glass sinter which communicates with a free water surface by means of a hanging water column. By lowering the free water surface the distance between the sinter and the free water surface (i.e. the suction) increases. Beyond a critical value of the applied suction each increment of the suction leads to a decrease of the water content of the porous material. In the present paper, the pore volume is divided into a number of classes, each having the same volume.[5] The suction corresponding to the mid-point in each volume class is substituted into the capillary-rise equation to obtain the pore radius for that class. In a similar way, mercury-injection data and liquid-nitrogen desorption isotherms are used to define the volume-radius characteristics of porous materials.

Assume there are m classes of pores in the porous medium and that each class occupies the same proportion of the total porosity.[5] The interacting areas of these classes in section are denoted by $a_1, a_2, a_3, \ldots$, and the radii characterizing these pore classes are $r_1 > r_2 > r_3 \ldots$. Flow will be determined by pore interactions and, for Poiseuille flow, both area and radius interactions will contribute. The resistance to flow in a pore sequence is determined by the square of the smaller pore radius of each interaction. Thus the permeability is given by

$$K = \frac{1}{8}\begin{bmatrix} a_1a_1r_1^2 & a_1a_2r_2^2 & a_1a_3r_3^2 & a_1a_4r_4^2 & \cdots & a_1a_mr_m^2 \\ a_2a_1r_2^2 & a_2a_2r_2^2 & a_2a_3r_3^2 & a_2a_4r_4^2 & \cdots & a_2a_mr_m^2 \\ a_3a_1r_3^2 & a_3a_2r_3^2 & a_3a_3r_3^2 & a_3a_4r_4^2 & \cdots & a_3a_mr_m^2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_ma_1r_m^2 & a_ma_2r_m^2 & a_ma_3r_m^2 & a_ma_4r_m^2 & \cdots & a_ma_mr_m^2 \end{bmatrix}$$

but since

$$a_1 = a_2 \ldots a_m = \varepsilon^{\frac{2}{3}}/m,$$

then

$$K = \tfrac{1}{8}\varepsilon^{\frac{4}{3}}m^{-2}(r_1^2 + 3r_2^2 + 5r_3^2 \ldots + (2m-1)r_m^2). \qquad (1)$$

For the computation of unsaturated permeability the value of ε taken is that of the liquid-filled pore space and the r^2 series is commenced at the appropriate pore class, that is the largest pore containing liquid.

Rather than introduce discontinuities into the pore-size distribution, it is preferable to use the true continuous variation in pore size with variation in volume porosity.

Thus, when

$$\varepsilon(r) = \text{volume of pores per unit volume with radius } < r;$$
$$\varepsilon(R) = \varepsilon = \text{porosity, } R \text{ is maximum pore radius};$$
$$\eta(r) = \varepsilon - \varepsilon(r) = \text{area of pores per unit cross-section with radius } \geqslant r,$$

the permeability is given by

$$K = (2/\varepsilon^{\frac{2}{3}})\int_0^R \eta'(r)\mathrm{d}r \int_0^r e^2\eta'(e)\mathrm{d}e,$$

and integrating by parts

$$\int_0^R \eta'(r)\mathrm{d}r \int_0^r e^2\eta'(e)\mathrm{d}e = \left[\eta(r)\int_0^r e^2\eta'(e)\mathrm{d}e\right]_0^R - \int_0^R r^2\eta(r)\eta'(r)\mathrm{d}r$$
$$= \tfrac{1}{2}\int_0^{\varepsilon^2} r^2\mathrm{d}(\eta)^2;$$

if the distribution is discontinuous, this is to be interpreted as a Stieltjes integral. If the Poiseuille coefficient is included, then

$$K = \frac{1}{8}\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\int_0^{\varepsilon^2} r^2\mathrm{d}(\eta)^2,$$

and $\varepsilon^{\frac{4}{3}}/\varepsilon^2$ may be regarded as a normalizing function. For saturated flow this is similar to Childs and Collis-George [4] except that $M\varepsilon^2$ is replaced by $1/8\varepsilon^{\frac{2}{3}}$.

The saturated permeability can be obtained by plotting $(\varepsilon-\varepsilon_r)^2$ against r^2 where r is the radius equivalent to the applied suction when the liquid-filled porosity is ε_r. The area under this curve is substituted in the following equation to obtain the permeability:

$$K = \frac{1}{8}\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\int_0^{\varepsilon^2} \mathrm{d}(\varepsilon-\varepsilon_r)^2 r^2;$$

alternatively,

$$K = \frac{1}{8}\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}2\int_0^{\varepsilon} \mathrm{d}(\varepsilon-\varepsilon_r)r^2(\varepsilon-\varepsilon_r).$$

Applicability of this method for obtaining steady-state permeability depends on factors, such as isotropy of the total pore space distribution in the medium, meaningful pore-size distribution data, and random distribution of the pore-size classes. The method will not give satisfactory results if the medium has been compressed to produce a laminated effect. The material must be inert in terms of swelling and shrinkage response to the liquid used in obtaining pore-size distribution data and permeability measurements. Further, special consideration should be given to contact angle between liquid and the porous medium.[8] Completely random dispersion of the pore-size classes is assumed in the above derivation and where aggregation of pores of similar size occurs, this restriction of pore continuity will affect the coefficients of the r^2 series in 1. Clustering of pores of similar radii is probably the most important factor leading to deviations between expected and actual permeabilities of inert porous media.

UNCONSOLIDATED MEDIA

Childs and Collis-George [4] presented data for three unconsolidated materials and a comparison of computed and experimentally determined permeabilities is shown in fig. 1*a*, 1*b* and 1*c*. Similar data of Youngs [4] are shown in fig. 1*d* and 1*e*. The computed curves for permeability in relation to water content obtained by the method of Childs and Collis-George, and Marshall, are shown for comparison.

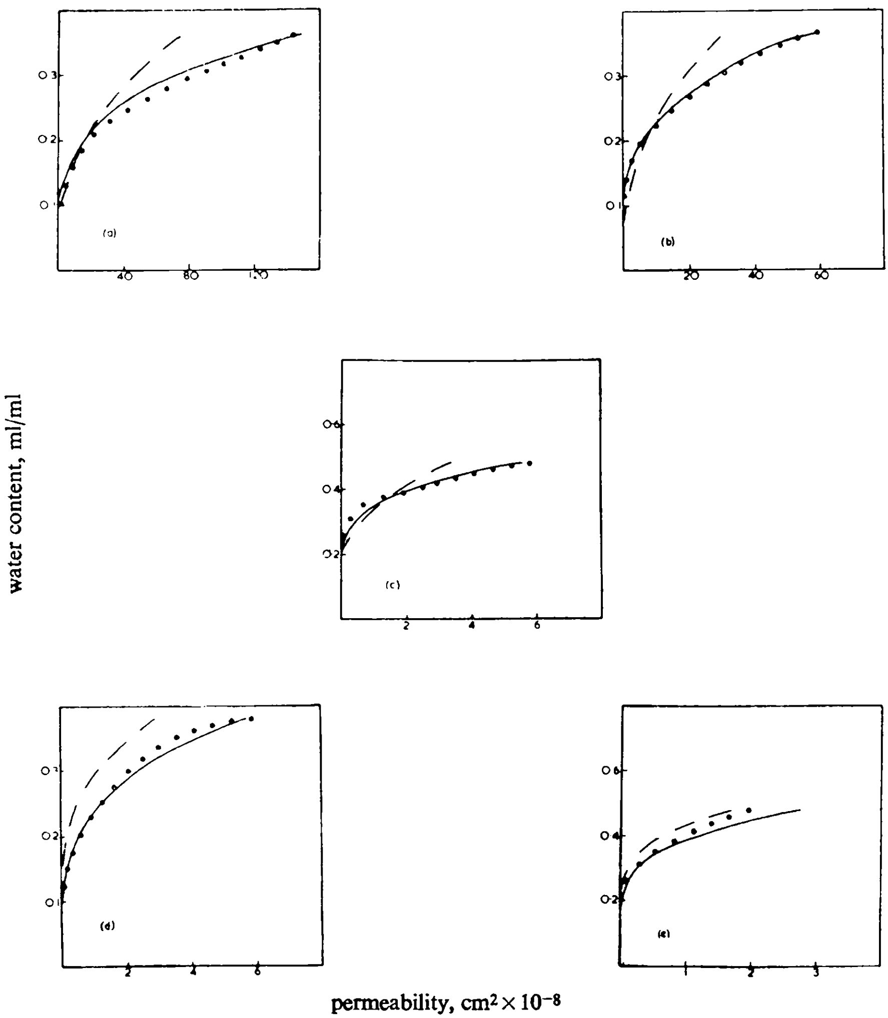

FIG. 1.—The relation between permeability and water content. - - - - theor.;[4, 5] —— present theory; . . . . . expt. *a*, *b* and *c*;[4] *d* and *e*.[9]

In this latter method of computation, the " matching factor " used by Childs and Collis-George has been discarded and the pore radius together with the Poiseuille coefficient 1/8 are used; the pore radius replaces the reciprocal of capillary suction. It is clear that with these types of media the present method provides a satisfactory interpretation of the saturated and unsaturated liquid permeability in terms of porosity, liquid-filled porosity and pore size distribution and is a marked improvement on previous methods of computation.

This conclusion is reinforced when experimental results obtained by Carman [10, 11] are considered. The results he obtained [10] for the pore-volume distribution of a Linde silica plug have been used to estimate the permeability of the plug. As with the other calculations, the pore radius-volume curve was divided into ten equal volume classes and the radius at the mid-point of each class was obtained. If the monolayer is assumed to be immobile it is necessary to subtract the monolayer thickness of 6 Å from the radii for each pore class; in addition, the value substituted for the porosity in eqn. (1) is $\varepsilon = 0{\cdot}402$ and this is obtained by subtracting the volume occupied by the monolayer from the total void volume ($\varepsilon = 0{\cdot}506$). When the intrinsic permeability for the Linde silica plug is calculated in this way, a value of 3×10^{-15} cm^2 is obtained and this compares with the permeability of $3{\cdot}1 \times 10^{-15}$ obtained for a similar plug.[11] The experimental values for saturated permeability obtained by Childs and Collis-George,[4] Youngs,[9] and Klute and Wilkinson,[12] are presented in fig. 2*c*. From this figure and the results obtained with the Linde silica plug it can be seen that the proposed method of computation satisfactorily describes the permeability of unconsolidated porous media over the range $3{\cdot}1 \times 10^{-15}$ to $1{\cdot}5 \times 10^{-6}$ cm^2.

RELATIVE PERMEABILITY

Several workers have shown similarity between relative permeabilities of a number of materials. All the materials showing almost identical courses of relative permeability are characterized by very narrow pore size distributions and may be regarded as consisting of a single pore-size class only.

Thus,

$$K_{\text{sat}} = \tfrac{1}{8}\varepsilon^{\frac{4}{3}}m^{-2}m^2r^2,$$

and

$$K_{\text{unsat}} = \tfrac{1}{8}\varepsilon_u^{\frac{4}{3}}m^{-2}n^2r^2,$$

where ε_u is the liquid-filled pore space, n is the number of liquid-filled pore classes and m is the total number of pore classes:

$$\varepsilon_u = n\varepsilon/m.$$

Thus,

$$K_{\text{unsat}}/K_{\text{sat}} = \varepsilon_u^{\frac{4}{3}}\varepsilon^{-\frac{4}{3}}m^{-2}n^2$$
$$= \varepsilon_u^{\frac{10}{3}}\varepsilon^{-\frac{10}{3}} \tag{3}$$

A comparison of data given by Carman [11] with those obtained by this method of computation is given in table 1. The agreement between the actual and computed relative permeabilities is most satisfactory, and discrepancies can be attributed to variations in total porosity and deviations from the assumed single-pore radius class. For a medium having a Gaussian pore-size distribution, (3) would lead to underestimates of relative permeability at high degrees of saturation and over-estimates at low degrees of saturation.

TABLE 1.—RELATIVE PERMEABILITY

% saturation	Carman	Wyckoff and Botset	Leverett	Childs and Collis-George	mean	computed—3 $\varepsilon = 0{\cdot}40$	computed—3 $\varepsilon = 0{\cdot}50$
100	1·00	1·00	1·00	1·00	1·00	1·000	1·000
90	0·66	0·70	0·76	0·78	0·725	0·716	0·700
80	0·44	0·47	0·56	0·53	0·500	0·483	0·475
70	0·27	0·29	0·35	0·30	0·302	0·310	0·305
60	0·16	0·16	0·21	0·17	0·175	0·185	0·182
50	0·087	0·09	0·11	0·082	0·092	0·101	0·092
40	0·040	0·04	0·04	0·042	0·041	0·0485	0·042
30	0·022	0·02	(0)	0·014	0·014	0·0183	0·018

CONSOLIDATED MEDIA

Eqn. (1) can be regarded as describing the saturated and unsaturated permeability of unconsolidated media and in this sense these media can be regarded as ideal systems. It is necessary to consider the characteristics of consolidated media which would cause deviations from ideal behaviour. In oil reservoir technology, flow problems relate to " consolidated " porous solids which are, in general, sedimentary deposits, such as sandstones and limestones. These materials when first deposited were probably laminated but showed reasonable isotropy within the larger laminae. On a particle-size basis, it would be expected that the parent deposits of sandstones would have a lower initial porosity than the more finely divided calcareous materials.

For some materials, Wyllie and Spangler [2] have shown that in the size of samples commonly used in their various measurements of formation factor and permeability there is not, in fact, pronounced anisotropy. The formation factor is defined as the ratio of resistance of the medium saturated with a conducting fluid to the resistance of the fluid. On their evidence it could be assumed that if consolidation has been accompanied by downward compression of the beds, in most instances this compression has not resulted in appreciable anisotropy. Cementation of the parent materials, either by deposition of finely divided matter or by precipitation of soluble substances within the porous matrix, would seem to be the more significant process in reducing the parent-bed porosity to that of the consolidated bed.

There are a number of reasons, to be discussed subsequently, which would suggest that the present theory would be of limited value for computing permeability of consolidated media. Nevertheless, the permeability values of various workers [3, 13, 14] plotted in fig. 2*a* and fig. 2*b* indicate that the agreement is good. Both positive and negative deviations from the expected permeability are encountered but there is a tendency for the computed values to over-estimate. These deviations are not related to the porosity of the medium. In fig. 2*c*, the data for unconsolidated media are collected and permeabilities of a few consolidated media ($K > 1 \times 10^{-8}$ cm^2) are included for comparison. The data in fig. 2 drawn from Faris *et al.*[3] have been corrected for Knudsen flow but the other gas-permeability data have not been corrected. In all the calculations of the permeability of of consolidated media reported here, the pore radius has been based on the pore-size distribution as determined by the mercury-injection method. This method appears to be the most satisfactory one since Joyner *et al.*[15] have obtained close agreement between pore-size distribution derived from liquid-nitrogen desorption

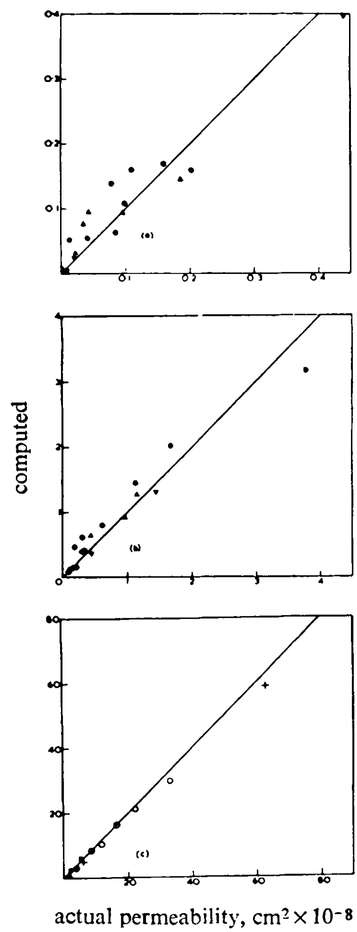

FIG. 2.—Comparison of computed and measured permeability.

▽ Burdine *et al.*;[13] × Childs and Collis-George;[4] ● Faris *et al.*;[3] ○ Klute and Wilkinson;[12] ▽ Purcell;[14] □ Youngs[9] isotherms and mercury-injection data. Purcell[14] has presented both mercury-injection data and results for water desorption in a centrifugal field. Using the water-desorption results, Marshall has obtained good agreement between calculated and experimentally determined gas permeability. However, the centrifuge method is probably the least satisfactory one for obtaining pore-size distribution data.

When the present method was used to calculate the permeability of Bradford sandstone ($\varepsilon = 0{\cdot}148$), the computed value $7{\cdot}8 \times 10^{-11}$ cm^2 is much larger than the experimental value[2] of $2{\cdot}8 \times 10^{-11}$ cm^2. This and other discrepancies probably arise from the nature of the cementation process.

With cemented (consolidated) materials the process of cementation may occur in a number of ways, some of which are given below:

(i) selective and complete occupation of fine pore classes;

(ii) selective and complete occupation of coarse pore classes;

(iii) selective and complete occupation of a number of pore classes of any size;

(iv) random occupation of pores, that is, incomplete occupation of a number of pore size classes;

(v) uniform, but not complete, occupation of the pores of all pore classes, that is, the same volume proportion of individual pores has been filled with cementing material.

If, for simplicity, we assume that the initial bed was made up of pores of almost equal radii, then for materials cemented by methods (i) to (iv), it would be appropriate to compute permeability thus:

$$K = \tfrac{1}{8}\varepsilon^{\frac{4}{3}} m_0^{-2} m_c^2 r_0^2, \qquad (4)$$

where ε is the initial porosity and is proportional to m_0; r_0 is the pore radius of the unconsolidated bed, ε_c is the porosity of the consolidated bed and is proportional to m_c and r_c is the pore radius of the consolidated bed.

Since in the initial condition the beds may be assumed to have had a porosity of about 0·40 ml/ml compared with approximately 0·20 ml/ml in the cemented condition, the above method would give values about $\frac{1}{4}$ those obtained by the method used to calculate the permeability values given in fig. 2*a* and 2*b*.

If $r_0 = r_c$, this method of calculation gives

$$K = \tfrac{1}{8}\varepsilon_c^{\frac{4}{3}} r_c^2. \qquad (5)$$

From this it might be inferred that cementation has occurred predominantly by method (5) and that the proportional decrease in porosity is accompanied by the same proportional decrease in radius, i.e.,

$$m_c^2/m_0^2 = r_c^2/r_0^2.$$

DISCUSSION

In "ideal" porous media, that is, in the absence of anisotropy, cementation or incomplete dispersion of pore sizes, the present model affords a method of computing with considerable accuracy both saturated and unsaturated permeability. Clearly in media where any of the above conditions, anisotropy, etc. do occur, both probability of continuity of the total pore area in section and the nature of the pore-radius interactions cannot be ascertained from measurements of porosity and pore-size distribution alone.

In such cases, measurement of the formation factor will provide the information expressing total pore area continuity. This same information is that required also in defining diffusive flow through porous solids as suggested by Millington,[16] so that for saturated unconsolidated systems,

$$D/D_0 = \varepsilon^{\frac{4}{3}} = 1/F.$$

However, the nature of the pore radius interactions will not be evident from measurements of formation factor and hence the value of the r^2 term for Poiseuille flow in these non-ideal systems. Faris *et al.*[3] have obtained good agreement between measured gas permeability and the permeability computed from the formation factor and a pore radius term which emphasizes the significance of small pores and in this respect resembles the r^2 series used here. Various methods have been used to obtain a " mean " radius of porous solids, but unless these methods take into account the probability of interaction of pores of differing radii, they must always remain of limited applicability for both Knudsen flow and Poiseuille permeability.

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